Commutativity of localized self-homotopy groups of symplectic groups

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Abstract

The self-homotopy group of a topological group G is the set of homotopy classes of self-maps of G equipped with the group structure inherited from G. We determine the set of primes p such that the p-localization of the self-homotopy group of $\mathrm{Sp}(n)$ is commutative. As a consequence, we see that this group detects the homotopy commutativity of p-localized $\mathrm{Sp}(n)$ by its commutativity almost all cases.

1 Introduction

For a group-like space G, the pointed homotopy set $[X, G]$ has a natural group structure inherited from G. We will always assume $[X, G]$ as a group with this group structure. This group has been studied for a long time, and there are many applications especially to the H-structure of G. See [1] and [9], for example. Put $X = G$. Then the group $[G, G]$ is called the self-homotopy group of G and denoted by $\mathcal{H}(G)$. The self homotopy group $\mathcal{H}(G)$ has also been studied extensively, especially, in connection with the H-structure of G, see [2], [12] and [11]. In particular, it is shown in [12] the following.

Theorem 1.1 (Kono and Ōshima [12]). *Let G be a compact, connected Lie group. Then $\mathcal{H}(G)$ is commutative if and only if G is isomorphic with T^n $(n \geq 0)$, $T^n \times \mathrm{Sp}(1)$ $(0 \leq n \leq 2)$ or $\mathrm{SO}(3)$, where T^n denotes the n-dimensional torus.*

Then we can say that for a connected Lie group G, $\mathcal{H}(G)$ reflects the homotopy commutativity of G to its commutativity effectively, since we have Hubbuck's torus theorem [8].

Localize at the prime p in the sense of [7]. Then it is an interesting problem to consider for a fixed G, how the H-structure of $G_{(p)}$ changes when we vary p. Kaji and the first named author obtained a result for a Lie group G when p is relatively large [9], [10]. Let us turn to the self homotopy group $\mathcal{H}(G)$. Let X be a finite complex, and let G be a path-connected group-like space. Then the group $[X, G]$ is known to be nilpotent, and then we can consider its localization $[X, G]_{(p)}$ at the prime p in the sense of [7]. Moreover, there is a natural isomorphism of groups:

$$[X, G]_{(p)} \cong [X_{(p)}, G_{(p)}]$$

See [7]. Then if G is a connected Lie group, it is also an interesting problem to consider how the group structure of $\mathcal{H}(G)_{(p)}$ changes if we vary p as is considered for $G_{(p)}$. Recently, Hamanaka and the second named author obtained:

Theorem 1.2 (Hamanaka and Kono [5]). *$\mathcal{H}(\mathrm{SU}(n))_{(p)}$ is commutative if and only if $p > 2n$ except for $n = 2$ and $(p, n) = (5, 3), (7, 4), (11, 6), (13, 7)$.*

As is shown in [13], $\mathrm{SU}(n)_{(p)}$ is homotopy commutative if and only if $p > 2n$. Then we can say that $\mathcal{H}(\mathrm{SU}(n))_{(p)}$ detects the homotopy commutativity of $\mathrm{SU}(n)_{(p)}$ very well.

The aim of this paper is to consider the above problem for $G = \mathrm{Sp}(n)$, and we will prove:

Theorem 1.3. *The group $\mathcal{H}(\mathrm{Sp}(n))_{(p)}$ is commutative if and only if $p > 4n$ except for $n = 1$ and $(p, n) = (3, 2), (5, 3), (7, 2), (11, 3), (19, 5), (23, 6)$.*

Since $\mathrm{Sp}(n)_{(p)}$ is homotopy commutative if and only if $p > 4n$ except for $(p, n) = (3, 2)$ by [13], we get:

Corollary 1.1. *$\mathrm{Sp}(n)_{(p)}$ is homotopy commutative if and only if $\mathcal{H}(\mathrm{Sp}(n))_{(p)}$ is commutative except for $n = 1$ and $(p, n) = (5, 3), (7, 2), (11, 3), (19, 5), (23, 6)$.*

Remark 1.1. Let p be an odd prime. As is well known [4], there is a homotopy equivalence $B\mathrm{Sp}(n)_{(p)} \simeq B\mathrm{Spin}(2n+1)_{(p)}$, and then, in particular, we have $\mathcal{H}(\mathrm{Sp}(n))_{(p)} \cong \mathcal{H}(\mathrm{Spin}(2n+1))_{(p)}$. Thus the above results for $\mathcal{H}(\mathrm{Sp}(n))_{(p)}$ implies those for $\mathcal{H}(\mathrm{Spin}(2n+1))_{(p)}$. We also have a similar result for $\mathcal{H}(\mathrm{Spin}(2n))_{(p)}$ when p is an odd prime [6].

2 Calculating commutators in the group $[X, \mathrm{Sp}(n)]$

Throughout this section, all spaces will be localized at the prime p.

Put $G_n = \mathrm{Sp}(n)$ and $X_n = G_\infty/G_n$. Let $q_k \in H^{4k}(BG_n; \mathbf{Z}_{(p)})$ be the k-th universal symplectic Pontrjagin class. Then the cohomology of G_n is given by

$$H^*(G_n; \mathbf{Z}_{(p)}) = \Lambda(x_3, x_7, \ldots, x_{4n-1}),\ x_{4k-1} = \sigma(q_k),$$

where σ is the cohomology suspension. We also have

$$H^*(X_n; \mathbf{Z}_{(p)}) = \Lambda(y_{4n+3}, y_{4n+7}, \ldots),\ \pi^*(x_i) = y_i$$

for the projection $\pi : G_\infty \to X_n$. Put $b_{4k+2} = \sigma(y_{4k+3}) \in H^*(\Omega X_n; \mathbf{Z}_{(p)})$ for $k \geq n$. We write a map $X \to K(\mathbf{Z}_{(p)}, k)$ corresponding to the cohomology class $x \in H^k(X; \mathbf{Z}_{(p)})$ by x, ambiguously. Then, in particular, since b_{4k+2} is a loop map, the map $b_{4k+2} : [X, \Omega X_n] \to H^{4k+2}(X; \mathbf{Z}_{(p)})$ is a homomorphism.

Now we recall from [15] how to determine the (non)triviality of commutators in the group $[X, G_n]$. Apply the functor $[X, -]$ to the fibre sequence

$$\Omega G_\infty \xrightarrow{\Omega\pi} \Omega X_n \xrightarrow{\delta} G_n \to G_\infty$$

in which all arrows are loop maps. Then we get an exact sequence of groups:

$$\widetilde{KSp}^{-2}(X)_{(p)} \xrightarrow{(\Omega\pi)_*} [X, \Omega X_n] \xrightarrow{\delta_*} [X, G_n] \to \widetilde{KSp}^{-1}(X)_{(p)} \tag{2.1}$$

Since $\widetilde{KSp}^{-1}(X)_{(p)}$ is abelian, commutators in $[X, G_n]$ are in the image of δ_*. We determine the (non)triviality of commutators in $[X, G_n]$ by the following proposition which is easily deduced by (2.1).

Proposition 2.1. *Let $\alpha, \beta \in [X, G_n]$, and put $\Phi = \bigoplus_{i=1}^k (b_{4n_i+2})_* : [X, \Omega X_n] \to \bigoplus_{i=1}^k H^{4n_i+2}(X; \mathbf{Z}_{(p)})$.*

1. *If there exists $\lambda \in [X, \Omega X_n]$ such that $\delta_*(\lambda) = [\alpha, \beta]$ and $\Phi(\lambda)$ is not in the image of $\Phi \circ (\Omega\pi)_*$, then $[\alpha, \beta]$ is not trivial.*

2. *Suppose that Φ is injective. Then $[\alpha, \beta]$ is not trivial if and only if there exists the above λ.*

In order to use Proposition 2.1, we need to describe $\lambda^*(b_{4m+2})$ explicitly, where λ is as in Proposition 2.1. In [15], it is shown that we can choose λ as:

Lemma 2.1. *For $\alpha, \beta \in [X, G_n]$, there exists $\lambda \in [X, \Omega X_n]$ such that $\delta_*(\lambda) = [\alpha, \beta]$ and for $k \geq n$,*

$$\lambda^*(b_{4k+2}) = \sum_{\substack{i+j=k+1 \\ 1 \leq i,j \leq n}} \alpha^*(x_{4i-1})\beta^*(x_{4j-1}).$$

We next describe $(\Omega\pi)_*(\xi)$ through the map $b_{4k+2} : [X, \Omega X_n] \to H^{4k+2}(X; \mathbf{Z}_{(p)})$ for $\xi \in \widetilde{KSp}^{-2}(X)_{(p)}$ to use Proposition 2.1. Let $\mathbf{c}' : G_n \to \mathrm{U}(2n)$ denote the complexification map. We also denote the complexification $\widetilde{KSp}^*(X)_{(p)} \to \widetilde{K}^*(X)_{(p)}$ by $\mathbf{c}'$. Let ch_k denote the $2k$-dimensional part of the Chern character.

Lemma 2.2. *For $\xi \in \widetilde{KSp}^{-2}(X)_{(p)}$, we have*

$$(b_{4k+2} \circ \Omega\pi)_*(\xi) = (-1)^{k+1}(2k+1)!\mathrm{ch}_{2k+1}(\mathbf{c}'(\xi)).$$

Proof. Let c_k be the k-th universal Chern class. Then we have $\mathbf{c}'(c_{2k}) = (-1)^k q_k$, and thus

$$(b_{4k+2} \circ \Omega\pi)_*(\xi) = \sigma^2(q_{k+1})(\xi) = (-1)^{k+1}\mathbf{c}'(\sigma^2(c_{2k+2}))(\xi) = (-1)^{k+1}(2k+1)!\mathrm{ch}_{2k+1}(\mathbf{c}'(\xi)).$$

□

3 Proof of Theorem 1.3 for p odd

Throughout this section, we localize all spaces at the odd prime p unless otherwise is specified.

For a given positive integer n, let m be an arbitrary integer satisfying $m < n \leq 2m$. Let ϵ_{4k-1} be a generator of $\pi_{4k-1}(G_n) \cong \mathbf{Z}_{(p)}$ for $k \leq n$. Then we have

$$(\epsilon_{4k-1})^*(x_{4k-1}) = \begin{cases} (2k-1)!u_{4k-1} & k \text{ is odd} \\ 2(2k-1)!u_{4k-1} & k \text{ is even} \end{cases} \tag{3.1}$$

where u_l is a generator of $H^l(S^l;\mathbf{Z}_{(p)})$. Define a map $\theta : S^{4m-1}\times S^{4m+3} \to G_n$ by the composition

$$S^{4m-1} \times S^{4m+3} \xrightarrow{\epsilon_{4m-1}\times\epsilon_{4m+3}} G_n \times G_n \xrightarrow{\mu} G_n,$$

where μ is the multiplication of G_n. Then by (3.1), we have for $k < l$:

$$\theta^*(x_{4k-1}x_{4l-1}) = \begin{cases} 2(2m-1)!(2m+1)!u_{4m-1}\otimes u_{4m+1} & (k,l)=(m,m+1) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Let $j\colon G_n \to G_{2m}$ be the inclusion, and let $\psi^2 : BG_n \to BG_n$ be the unstable Adams operation of degree 2 [18]. We consider the commutator $[j\circ\Omega\psi^2, j]$ in $[G_n, G_{2m}]$ by pulling back to $S^{4m-1}\times S^{4m+3}$ through θ. By Lemma 2.1, there exists $\lambda \in [G_n, \Omega X_{2m}]$ such that $\delta_*(\lambda) = [j\circ\Omega\psi^2, j]$ and

$$\lambda^*(b_{8m+2}) = \sum_{\substack{i+j=n+1\\ 1\le i,j\le n}} (\Omega\psi^2)^*(x_{4i-1})x_{4j-1}.$$

By definition of ψ^2, we have $(\Omega\psi^2)^*(x_{4k-1}) = 2^{2k}x_{4k-1}$. Then we get

$$\lambda^*(b_{8m+2}) = \sum_{\substack{i+j=n+1\\ 1\le i,j\le n}} 2^{2i}x_{4i-1}x_{4j-1}$$

and thus by (3.2),

$$\theta^*\circ\lambda^*(b_{8m+2}) = 2^{2m}(-3)(2m-1)!(2m+1)!u_{4m-1}\otimes u_{4m+1}.$$

On the other hand, we have $\widetilde{KSp}^{-2}(S^{4m-1}\times S^{4m+3})_{(p)} \cong \mathbf{Z}_{(p)}$ and its generator ξ can be chosen to satisfy

$$\mathrm{ch}_{4m+1}(\mathbf{c}'(\xi)) = (4m+1)!u_{4m-1}\otimes u_{4m+3}.$$

If we see that $\theta^*\circ\lambda^*(b_{8m+2})$ is not in the $\mathbf{Z}_{(p)}$-module generated by $(4m+1)!u_{4m-1}\otimes u_{4m+3}$, by Proposition 2.1, we can conclude that $\theta^*([j\circ\Omega\psi^2, j]) = j\circ[\Omega\psi^2, 1_{G_n}]\circ\theta$ is non-trivial which implies $\mathcal{H}(G_n)$ is not commutative. Put m as in the following table. Then we can easily see that m satisfies $m < n \le 2m$ and $\frac{(4m+1)!}{(2m-1)!(2m+1)!} = (4m+1)\binom{4m}{2m-1} \equiv 0\ (p)$ by Lucas' formula, and thus $\mathcal{H}(G_n)$ is not commutative in these cases.

$p<n$	$m\equiv 0\ (p),\ 0<n-m\le p$
$p=n$	$m=p-1$
$n<p<n+3\ (p\ge 13)$	$m=p-3$
$n+3\le p<2n$	$m=\frac{p-3}{2}$
$2n<p<4n-1\ (p\equiv -1\ (4))$	$m=\frac{p+1}{4}$
$2n<p<4n-1\ (p\equiv 1\ (4),\ p>5)$	$m=\frac{p+3}{4}$
$(p,n)=(5,2)$	$m=1$
$(p,n)=(7,6)$	$m=5$
$(p,n)=(11,9),(11,10)$	$m=8$

Recall from [13] that G_n is homotopy commutative if $p > 4n$ or $(p, n) = (3, 2)$ which implies $\mathcal{H}(G_n)$ is commutative for $p > 4n$ or $(p, n) = (3, 2)$. Then the remaining cases are:

1. $p = 4n - 1$
2. $(p, n) = (7, 5)$
3. $(p, n) = (5, 4)$
4. $(p, n) = (5, 3)$

3.1 Case 1

In this case we have a homotopy equivalence [14] $G_n \simeq \prod_{k=1}^{n} S^{4k-1}$. Assume $n \geq 14$. We define $\alpha \in \mathcal{H}(G_n)$ by the composite

$$G_n \xrightarrow{\rho} S^3 \times S^7 \times S^{11} \times S^{15} \times S^{4n-37} \xrightarrow{q} S^{4n-1} \xrightarrow{\epsilon_{4n-1}} G_n,$$

where ρ is the projection and q is the pinch map onto the top cell. We also define $\beta \in \mathcal{H}(G_n)$ by

$$G_n \xrightarrow{\rho'} S^{4n-1} \xrightarrow{\epsilon_{4n-1}} G_n,$$

where ρ' is the projection. Then we have

$$[\alpha, \beta] = \gamma \circ (\epsilon_{4n-1} \times \epsilon_{4n-1}) \circ ((q \circ \rho) \times \rho') \circ \Delta,$$

where $\gamma : G_n \times G_n \to G_n$ and $\Delta : G_n \to G_n \times G_n$ denote the commutator map of G_n and the diagonal map, respectively. Now one can easily see $((q \circ \rho) \times \rho') \circ \Delta$ induces an injection $[S^{4n-1} \times S^{4n-1}, G_n] \to \mathcal{H}(G_n)$. On the other hand, we have $\gamma \circ (\epsilon_{4n-1} \times \epsilon_{4n-1}) = \langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle \circ q'$, where $q' : S^{4n-1} \times S^{4n-1} \to S^{8n-2}$ is the pinch map onto the top cell and $\langle -, - \rangle$ means a Samelson product. Then since q' induces an injection $\pi_{8n-2}(G_n) \to [S^{4n-1} \times S^{4n-1}, G_n]$ and the Samelson product $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle \in \pi_{8n-2}(G_n)$ is non-trivial by [3], we obtain that the commutator $[\alpha, \beta]$ is non-trivial. Thus $\mathcal{H}(G_n)$ is not commutative.

We next assume $8 \leq n \leq 13$. By looking at the homotopy groups of spheres [16], the above Samelson product $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle$ factors as $\langle \epsilon_{4n-1}, \epsilon_{4n-1} \rangle = i \circ \alpha_1(3)$, where $i : S^3 \to G_n$ is the inclusion and $\alpha_1(2k-1)$ is a generator of $\pi_{2k+2p-4}(S^{2k-1}) \cong \mathbf{Z}/p$. Put $X = S^3 \times S^7 \times S^{11} \times S^{4n-13} \times S^{4n-9} \times S^{4n-5}$. We define $\alpha, \beta \in \mathcal{H}(G_n)$ by

$$G_n \xrightarrow{\rho} X \xrightarrow{q} S^{3p-3} \xrightarrow{\alpha_1(p)} S^p \xrightarrow{\epsilon_p} G_n$$

and

$$G_n \xrightarrow{\rho'} S^p \xrightarrow{\epsilon_n} G_n,$$

respectively, where ρ and ρ' are the projections and q is the pinch map onto the top cell. Then we get

$$[\alpha, \beta] = i \circ \alpha_1(3) \circ \alpha_1(2p) \circ q' \circ ((q \circ \rho) \times \rho') \circ \Delta,$$

where $q' : S^{3p-3} \times S^p \to S^{4p-3}$ is the pinch map. As is seen above, the maps i and $q' \circ ((q \circ \rho) \times \rho') \circ \Delta$ induce injections $\pi_{4p-3}(S^3) \to \pi_{4p-3}(G_n)$ and $\pi_{4p-3}(G_n) \to \mathcal{H}(G_n)$, respectively. Since $\alpha_1(3) \circ \alpha_1(2p) \neq 0$ as in [16], we obtain that the commutator $[\alpha, \beta]$ is non-trivial. Thus $\mathcal{H}(G_n)$ is not commutative.

For $n \leq 7$, the case 1 occurs only when $n = 1, 2, 3, 5, 6$. We only prove the case $n = 6$ since the remaining cases are quite similarly proved. Note that for $n = 6$ in the case 1, we have $p = 23$. One can easily see that the dimension of cells of $G_6 / \bigvee_{k=1}^{6} S^{4k-1}$ is in the set $I = \{0\} \cup \bigcup_{k=2}^{6} \{4(n_1 + \cdots + n_k) - k \mid 1 \leq n_1 < \cdots < n_k \leq 6\}$. On the other hand, Since $G_6 \simeq \prod_{k=1}^{6} S^{4k-1}$, we see that the homotopy groups of G_6 in dimension $k \in I$ for all $k \in I$ are trivial by looking at the homotopy groups of spheres [16]. Then the inclusion $\bigvee_{k=1}^{6} S^{4k-1} \to G_6$ induces an injection $\mathcal{H}(G_6) \to \bigoplus_{k=1}^{6} \pi_{4k-1}(G_6)$, and so $\mathcal{H}(G_6)$ is commutative.

3.2 Case 2

In this case, we have $G_5 \simeq B_1 \times B_2 \times S^{11}$, where B_k is an S^{4k-1}-bundle over S^{4k+11} for $k = 1, 2$, see [14]. We first calculate $K^*(G_5)_{(7)}$. Note that $K^*(B_k)$ for $k = 1, 2$ and $K^*(S^{11})_{(7)}$ are free $\mathbf{Z}_{(7)}$-module, we have

$$K^*(G_5)_{(7)} \cong K^*(B_1)_{(7)} \otimes K^*(B_2)_{(7)} \otimes K^*(S^{11})_{(7)}.$$

Let A_k be the $(4k + 11)$-skeleton of B_k for $k = 1, 2$. Then we have $A_2 \simeq \Sigma^4 A_1$.

Let u' be the composite of the inclusions $\Sigma A_1 \to \Sigma G_5 \to BG_5 \to B\mathrm{U}(\infty)$. Since A_1 is a retract of $\Sigma \mathbf{C}P^7$, we get $\mathrm{ch}(u') = \Sigma t_3 + \frac{1}{7!}\Sigma t_{15}$ where t_3, t_{15} are generators of $\widetilde{H}^*(A_1; \mathbf{Z}_{(7)})$ with $|t_k| = k$ and Σ stands for the suspension isomorphism. Let v' be the composite of the pinch map $\Sigma A_1 \to S^{16}$ and a generator of $\pi_{16}(B\mathrm{U}(\infty)) \cong \mathbf{Z}_{(7)}$. Then we see $\mathrm{ch}(v') = \Sigma t_{15}$ by choosing a suitable generator of $\pi_{16}(B\mathrm{U}(\infty))$. Consider the exact sequence

$$0 \to \widetilde{K}^{-1}(S^{15})_{(7)} \to \widetilde{K}^{-1}(A_1)_{(7)} \to \widetilde{K}^{-1}(S^3)_{(7)} \to 0$$

induced from the cofibre sequence $S^3 \to A_1 \to S^{15}$. Then we get $\widetilde{K}^{-1}(A_1)_{(7)}$ is generated by u' and v'. Since the inclusion $A_k \to B_k$ induces an isomorphism $\widetilde{K}^{-1}(B_k)_{(7)} \to \widetilde{K}^{-1}(A_k)_{(7)}$, we get

$$K^*(G_5)_{(7)} = \Lambda(u_1, u_2, v_1, v_2, w), \ |u_k| = |v_k| = |w| = -1$$

such that for $k = 1, 2$,

$$\mathrm{ch}(u_k) = \Sigma x_{4k-1} + \frac{1}{7!}\Sigma x_{4k+11}, \ \mathrm{ch}(v_k) = \Sigma x_{4k+11}, \ \mathrm{ch}(w) = \Sigma x_{11}.$$

Since $\mathbf{q} \circ \mathbf{c}' = 2$ for the quaternionization $\mathbf{q} : K^*(G_5)_{(7)} \to KSp^*(G_5)_{(7)}$, we obtain:

Lemma 3.1. *$\widetilde{KSp}^{-2}(G_5)_{(7)}$ is a free $\mathbf{Z}_{(7)}$-module with a basis $\{a_1, \ldots, a_{10}\}$ such that*

$$\mathrm{ch}_{15}(\mathbf{c}'(a_k)) = \begin{cases} \frac{1}{7}x_{11}x_{19} & k = 1 \\ x_{11}x_{19} & k = 2 \\ 0 & k \neq 1, 2. \end{cases}$$

Let α be the composite of the projection $G_5 \to B_2$ and the inclusion $B_2 \to G_5$. We consider the commutator $[1_{G_5}, \alpha]$. By Lemma 2.1, there exists $\lambda \in [G_5, \Omega X_5]$ such that $\delta_*(\lambda) = [1_{G_5}, \alpha]$ and $\lambda^*(b_{30}) = x_{11}x_{19}$. On the other hand, it follows from Lemma 2.2 and Lemma 3.1 that the image of the map $b_{30} \circ (\Omega\pi)_* : \widetilde{KSp}^{-2}(G_5)_{(7)} \to H^{30}(G_5; \mathbf{Z}_{(7)})$ is generated by $7x_{11}x_{19}$. Then by Proposition 2.1, we conclude that $[1_{G_5}, \alpha]$ is non-trivial which implies $\mathcal{H}(G_5)$ is not commutative.

3.3 Case 3

In this case, we have a homotopy equivalence $G_4 \cong B_1 \times B_2$ where B_k is an S^{4k-1}-bundle over S^{4k+7} for $k = 1, 2$ [14]. As in the previous case, we have

$$K^*(G_4)_{(5)} = \Lambda(u_1, u_2, v_1, v_2), \ |u_k| = |v_k| = -1$$

such that for $k = 1, 2$,

$$\mathrm{ch}(u_k) = \Sigma x_{4k-1} + \frac{1}{5!}\Sigma x_{4k+7}, \ \mathrm{ch}(v_k) = \Sigma x_{4k+7},$$

and thus we obtain:

Lemma 3.2. $\widetilde{KSp}^{-2}(G_4)_{(5)}$ *is a free* $\mathbf{Z}_{(5)}$*-module with a basis* $\{a_1, \ldots, a_6\}$ *such that*

$$\mathrm{ch}_{11}(\mathbf{c}'(a_k)) = \begin{cases} \frac{1}{5}x_7x_{15} & k = 1 \\ x_7x_{15} & k = 2 \\ 0 & k \neq 1, 2. \end{cases}$$

Let $\psi^2 : BG_4 \to BG_4$ be the unstable Adams operation of degree 2 as above. We consider $[\Omega\psi^2, 1_{G_4}]$. By Lemma 2.1, there exists $\lambda \in [G_4, \Omega X_4]$ such that $\delta_*(\lambda) = [\Omega\psi^2, 1_{G_4}]$ and

$$\lambda^*(b_{22}) = 2^4x_7x_{15} + 2^8x_{15}x_7 = 2^4 \cdot 3 \cdot 5x_7x_{15}.$$

Then by Lemma 2.2 and Lemma 3.2, we see that $\lambda^*(b_{22})$ is not in the image of $b_{22} \circ (\Omega\pi)_*$. Then by Proposition 2.1, we obtain $[\Omega\psi^2, 1_{G_4}]$ is not trivial, and thus $\mathcal{H}(G_4)$ is not commutative.

3.4 Case 4

This case is very special. We first show:

Lemma 3.3. *The map* $(b_{14} \times b_{18})_* : [G_3, \Omega X_3] \to H^{14}(G_3; \mathbf{Z}_{(5)}) \oplus H^{18}(G_3; \mathbf{Z}_{(5)})$ *is injective.*

Proof. Note that the 23-skeleton of X_3 is $A = S^{15} \cup e^{19} \cup e^{23}$. Then since G_3 is of dimension 21, the inclusion $A \to X_3$ induces an isomorphism of groups $[G_4, \Omega A] \xrightarrow{\cong} [G_4, \Omega X_3]$. Since for $k \leq 23$, $\pi_k(A)$ is in the stable range. Then one can easily see that

$$\pi_k(A) \cong \begin{cases} \mathbf{Z}_{(5)} & k = 15, 19, 23 \\ 0 & k \neq 15, 19, 23 \text{ and } k \leq 23. \end{cases}$$

Thus we can easily deduce that $[G_3, \Omega X_3]$ is a free $\mathbf{Z}_{(5)}$-module. On the other hand, the rationalization of the map $(b_{14} \times b_{18})_*$ is injective. Then the proof is completed. $\square$

As in the case 2, we have

$$K^*(G_3)_{(5)} = \Lambda(u, v, w),\ |u| = |v| = |w| = -1$$

such that

$$\mathrm{ch}(u) = \Sigma x_3 + \frac{1}{5!}\Sigma x_{11},\ \mathrm{ch}(v) = \Sigma x_{11},\ \mathrm{ch}(w) = \Sigma x_7.$$

Then we get $\widetilde{KSp}^{-2}(G_3)_{(5)}$ is a free $\mathbf{Z}_{(5)}$-module with a basis $\{a_1, a_2, a_3\}$ such that

$$\mathrm{ch}(\mathbf{c}'(a_1)) = x_3x_{11},\ \mathrm{ch}(\mathbf{c}'(a_2)) = \frac{1}{5}x_7x_{11},\ \mathrm{ch}(\mathbf{c}'(a_3)) = x_7x_{11}.$$

Thus we obtain:

Lemma 3.4. *The image of* $(b_{14} \times b_{18})_* \circ (\Omega\pi)_* : \widetilde{KSp}^{-2}(G_3)_{(5)} \to H^{14}(G_3; \mathbf{Z}_{(5)}) \oplus H^{18}(G_3; \mathbf{Z}_{(5)})$ *is generated by* $5x_3x_{11}$ *and* x_7x_{11}.

Let $\alpha, \beta \in \mathcal{H}(G_4)$. Then for a degree reason, we have $\alpha^*(x_{4k-1}) = \alpha_{4k-1}x_{4k-1}$ and $\beta^*(x_{4k-1}) = \beta_{4k-1}x_{4k-1}$, where $\alpha_i, \beta_i \in \mathbf{Z}_{(5)}$. Moreover, since $\mathcal{P}^1x_3 = x_{11}$, we have $\alpha_3 \equiv \alpha_{11},\ \beta_3 \equiv \beta_{11}\ (5)$. Let us consider the commutator $[\alpha, \beta]$. By Lemma 2.1, there exists $\lambda \in [G_3, \Omega X_3]$ such that $\delta_*(\lambda) = [\alpha, \beta]$ and

$$\lambda^*(b_{14}) = (\alpha_3\beta_{11} - \alpha_{11}\beta_3)x_3x_{11},\ \lambda^*(b_{18}) = (\alpha_7\beta_{11} - \alpha_{11}\beta_7)x_7x_{11}.$$

Since $\alpha_3\beta_{11} - \alpha_{11}\beta_3 \equiv 0\ (5)$, we obtain that $(b_{14} \times b_{18})_*(\lambda)$ is in the image of $(b_{14} \times b_{18})_* \circ (\Omega\pi)_*$ by Lemma 3.4. Thus by Proposition 2.1, $\mathcal{H}(G_3)$ is commutative.

4 Proof of Theorem 1.3 for $p = 2$

Throughout this section, spaces will be localized at the prime 2. We only consider $\mathcal{H}(G_n)$ for $n \geq 2$ since $\mathcal{H}(G_1)$ is obviously commutative.

For $m \geq 2$, put $N = 2^{m-2}$. Let $A = S^3 \cup e^7$ be the 7-skeleton of G_∞, and let $i : \Sigma A \to BG_\infty$ be the composite of inclusions $\Sigma A \to \Sigma G_\infty \to BG_\infty$. We write generators of $\widetilde{H}^*(A; \mathbf{Z}_{(2)})$ by t_3, t_7 where $|t_k| = k$. Then by [17], we can deduce

$$\mathrm{ch}(\mathbf{c}'(i)) = \Sigma u_3 - \frac{1}{6}\Sigma u_7. \tag{4.1}$$

For a generator $\beta_{\mathbf{R}}$ of $\widetilde{KO}(S^8)_{(2)}$, let $\bar{\alpha} : \Sigma^{8N-8}A \to G_\infty$ be the adjoint of $i \wedge \beta_{\mathbf{R}}^{N-1} : \Sigma^{8N-7}A \to BG_\infty$. Then by (4.1), we get

$$\bar{\alpha}^*(x_{8N-1}) = (4N-1)!\Sigma^{8N-7}\mathrm{ch}(\mathbf{c}'(i)) = -(4N-1)!\frac{1}{6}\Sigma^{8N-8}t_7.$$

Since the inclusion $G_{4N} \to G_\infty$ is an $(16N+2)$-equivalence and $\Sigma^{8N-8}A$ is of dimension $8N-1$, the map $\bar{\alpha} : \Sigma^{8N-8}A \to G_\infty$ factors as the composite of the map $\alpha : \Sigma^{8N-8}A \to G_{4N}$ and the inclusion $G_{4N} \to G_\infty$. In particular, we have

$$\alpha^*(x_{8N-1}) = -(4N-1)!\frac{1}{6}\Sigma^{8N-8}t_7.$$

Let ϵ be a generator of $\pi_{8N+3}(G_{4N})$. Then we get

$$\epsilon^*(x_{8N+3}) = (4N+1)!w,$$

where w denotes a generator of $H^{8N+3}(S^{8N+3};\mathbf{Z}_{(2)})$. Define a map $\theta : \Sigma^{8N-8}A \times S^{8N+3} \to G_{4N}$ by the composite

$$\Sigma^{8N-8}A \times S^{8N+3} \xrightarrow{\alpha\times\epsilon} G_{4N} \times G_{4N} \xrightarrow{\mu} G_{4N},$$

where μ is the multiplication of G_{4N}. Then by definition, we have:

$$\theta^*(x_{4k-1}) = \begin{cases} -(4N-1)!\frac{1}{6}\Sigma^{8N-8}t_7 \otimes 1 & k = 2N \\ (4N+1)!1 \otimes w & k = 2N+1 \\ 0 & k \neq 2N, 2N+1 \end{cases} \tag{4.2}$$

Consider the commutator $[\Omega\psi^3, 1_{G_{4N}}]$ in $\mathcal{H}(G_{4N})$ for the unstable Adams operation $\psi^3 : BG_{4N} \to BG_{4N}$ of degree 3. Then by Lemma 2.1, there exists $\lambda \in [G_{4N}, \Omega X_{4N}]$ such that

$$\lambda^*(b_{16N+2}) = \sum_{\substack{i+j=4N+1\\ 1\le i,j\le 4N}} (\Omega\psi^3)^*(x_{4i-1})x_{4j-1} = \sum_{\substack{i+j=4N+1\\ 1\le i,j\le 4N}} 3^{2i}x_{4i-1}x_{4j-1}.$$

Hence by (4.2), we get

$$\theta^* \circ \lambda^*(b_{16N+2}) = 3^{4N-1}\cdot 4(4N-1)!(4N+1)!\Sigma^{8N-8}t_7 \otimes w, \tag{4.3}$$

here $\delta_*(\lambda\circ\theta)$ equals to the commutator $[(\Omega\psi^3)\circ\theta, \theta]$ in $[\Sigma^{8N-8}A \times S^{8N+3}, G_{4N}]$.

In order to apply Proposition 2.1, we next calculate the free part of $\widetilde{KSp}^{-2}(\Sigma^{8N-8}A \times S^{8N+3})_{(2)}$. We know that the pinch map $q : \Sigma^{8N-8}A \times S^{8N+3} \to \Sigma^{16N-5}A$ induces an isomorphism between the free parts in $\widetilde{KSp}^{-2}_{(2)}$. Then we calculate $\widetilde{KSp}^{-2}(\Sigma^{16N-5}A)_{(2)}$. Consider the following commutative diagram of exact sequences induced from the cofibre sequence $S^{16N-2} \to \Sigma^{16N-5}A \to S^{16N+2}$.

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{KSp}^{-2}(S^{16N+2})_{(2)} \arrow[r] \arrow[d, "\mathbf{c}'=1"'] & \widetilde{KSp}^{-2}(\Sigma^{16N-5}A)_{(2)} \arrow[r] \arrow[d, "\mathbf{c}'"'] & \widetilde{KSp}^{-2}(S^{16N-2})_{(2)} \arrow[r] \arrow[d, "\mathbf{c}'=2"'] & 0 \\ 0 \arrow[r] & \widetilde{K}^{-2}(S^{16N+2})_{(2)} \arrow[r] & \widetilde{K}^{-2}(\Sigma^{16N-5}A)_{(2)} \arrow[r] & \widetilde{K}^{-2}(S^{16N-2})_{(2)} \arrow[r] & 0 \end{tikzcd}$$

Put $u' = \beta_c^{8N-2} \wedge \mathbf{c}'(i)$ and v' to be the complexification of the composite of the pinch map $\Sigma^{16N-3}A \to S^{16N+4}$ and a generator of $\pi_{16N+4}(B\mathrm{Sp}(\infty))$, where $\beta_{\mathbf{C}}$ is a generator of $\widetilde{K}^0(S^2)_{(2)}$. Then by (4.1), one sees that $\widetilde{K}^{-2}(\Sigma^{16N-5}A)_{(2)}$ is generated by u' and v' such that

$$\mathrm{ch}(u') = \Sigma^{16N-5}t_3 - \frac{1}{6}\Sigma^{16N-5}t_7,\ \mathrm{ch}(v') = \Sigma^{16N-5}t_7.$$

Put $u = \lambda\wedge i$ and v to be the composite of the pinch map $\Sigma^{16N-3}A \to S^{16N+4}$ and a generator of $\pi_{16N+4}(B\mathrm{Sp}(\infty))$, where λ is a generator of $\widetilde{KO}^0(S^{16N-4})_{(2)}$. Then by the above diagram, we obtain that $\widetilde{KSp}^{-2}(\Sigma^{16N-5}A)_{(2)}$ is a free $\mathbf{Z}_{(2)}$-module generated by u, v such that

$$\mathrm{ch}(\mathbf{c}'(u)) = 2\Sigma^{16N-5}t_3 - \frac{1}{3}\Sigma^{16N-5}t_7,\ \mathrm{ch}(\mathbf{c}'(v)) = \Sigma^{16N-5}t_7.$$

Summarizing, we get:

Lemma 4.1. *The free part of* $\widetilde{KSp}^{-2}(\Sigma^{8N-8}A \times S^{8N+3})_{(2)}$ *is generated by* $\bar{u}$ *and* $\bar{v}$ *such that*

$$\mathrm{ch}(\mathbf{c}'(\bar{u})) = 2\Sigma^{8N-8}t_3 \otimes w - \frac{1}{3}\Sigma^{8N-8}t_7 \otimes w, \ \mathrm{ch}(\mathbf{c}'(\bar{v})) = \Sigma^{8N-8}t_7 \otimes w.$$

For an integer k, we put $\nu_2(k) = m$ if $k = 2^m(2l-1)$. Then in general, we have

$$\nu_2(k!) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2^2} \right\rfloor + \left\lfloor \frac{k}{2^3} \right\rfloor + \cdots, \tag{4.4}$$

where $\lfloor x \rfloor = \max\{n \in \mathbf{Z} \mid n \leq x\}$.

Note that $H^{16N+2}(\Sigma^{8N-8}A \times S^{8N+3})$ is a free $\mathbf{Z}_{(2)}$-module. Then it follows from the above lemma, we obtain that the image of $b_{16N+2} \circ (\Omega\pi)_* : [\Sigma^{8N-8}A \times S^{8N+3}, \Omega X_{4N}] \to H^{16N+2}(\Sigma^{8N-8}A \times S^{8N+3}; \mathbf{Z}_{(2)})$ is generated by $(8N+1)!\Sigma^{8N-8}t_7 \otimes w$. It follows from (4.4) that $\nu_2((8N+1)!) = 2^{m+2} - 1$ and $\nu_2(4(4N-1)!(4N+1)!) = 2^{m+2} - m$. Then by Lemma 2.1, (4.3) and Lemma 4.1, we get that the commutator $[(\Omega\psi^3) \circ \theta, \theta]$ is non-trivial. If $N < n \leq 2N$, the map α and ϵ factors through the inclusion $j : G_n \to G_{4N}$, and so there exists a map $\hat{\theta} : \Sigma^{8N-8}A \times S^{8N+3} \to G_n$ such that $\theta = j \circ \hat{\theta}$. Then we obtain that $[(\Omega\phi^3) \circ \theta, \theta] = j \circ [\Omega\psi^3, 1_{G_n}] \circ \hat{\theta}$ is non-trivial which implies that $\mathcal{H}(G_n)$ is not commutative.

References

[1] M. Arkowitz and C.R. Crujel, *On maps of H-spaces*, Topology **6** (1967) 137-148.

[2] M. Arkowitz, H. Ōshima, and J. Strom, *Noncommutativity of the group of self homotopy classes of Lie groups*, Topology Appl. **125** (2002), no. 1, 87-96.

[3] R. Bott, *A note on the Samelson products in the classical groups*, Comment. Math. Helv. **34** (1960), 249-256.

[4] E.M. Friedlander, *Exceptional isogenies and the classifying spaces of simple Lie groups*, Ann. Math. (2) **101** (1975), 510-520.

[5] H. Hamanaka and A. Kono, *On the commutativity of the localized self homotopy groups of $SU(n)$*, to appear in Topology Appl.

[6] H. Hamanaka and A. Kono, *A note on the Samelson products in $\pi_*(\mathrm{SO}(2n))$ and the group* $[\mathrm{SO}(2n), \mathrm{SO}(2n)]$, Topology Appl. **154** (2007), no. 3, 567-572.

[7] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Mathematics Studies, No. 15. Notas de Matemática, No. 55. [Notes on Mathematics, No. 55] North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975.

[8] J. Hubbuck, *On homotopy commutative H-spaces*, Topology **8** (1969), 119-126.

[9] S. Kaji and D. Kishimoto, *Homotopy nilpotency in p-regular loop spaces*, Math. Z. **264** (2010), no. 1, 209-224.

[10] D. Kishimoto, *Homotopy nilpotency in localized* $\mathrm{SU}(n)$, Homology, Homotopy Appl. **11** (2009), no. 1, 61-79.

[11] D. Kishimoto and A. Kono, *On a conjecture of Ōhima*, Topology Appl. **156** (2009), no. 13, 2189-2192.

[12] A. Kono and H. Ōshima, *Commutativity of the group of self-homotopy classes of Lie groups*, Bull. London Math. Soc. **36** (2004), no. 1, 37-52.

[13] C.A. McGibbon, *Homotopy commutativity in localized groups*, Amer. J. Math. **106** (1984), no. 3, 665-687.

[14] M. Mimura, G. Nishida and H. Toda, *Mod p decomposition of compact Lie groups*, Publ. Res. Inst. Math. Sci. **13** (1977/78), no. 3, 627-680.

[15] T. Nagao, *On the groups* $[X, \mathrm{Sp}(n)]$ *with* $\dim X \leq 4n+2$, J. Math. Kyoto Univ. **48** (2008), no. 1, 149-165.

[16] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Studies **49**, Princeton Univ. Press, Princeton N.J., 1962.

[17] T. Watanabe, *Chern characters on compact Lie groups of low rank*, Osaka J. Math. **22** (1985), no. 3, 463-488.

[18] C. Wilkerson, *Self-maps of classifying spaces*, Lecture Notes in Math. **418**, Springer, Berlin, 1974.